Strong Edge Coloring of Maximum Degree 5 Planar Graphs

Seth Nelson

William & Mary

Joint work with Dr. Gexin Yu

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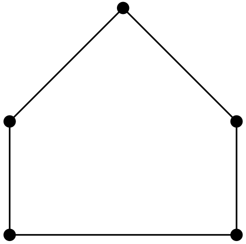

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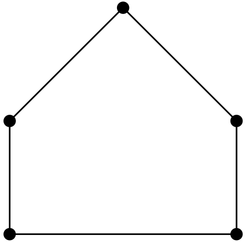
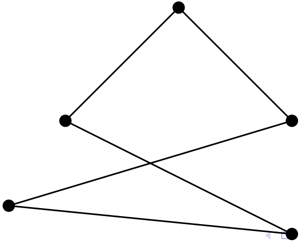


- Here it is again!



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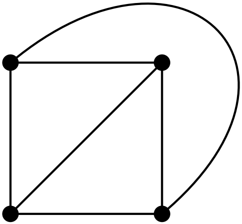
- Of course, there are other kinds of graphs.



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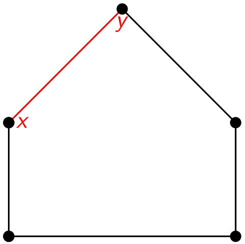

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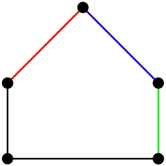

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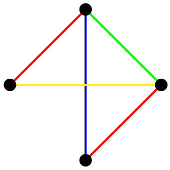

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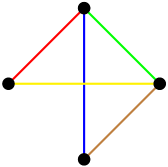

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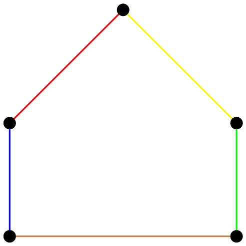

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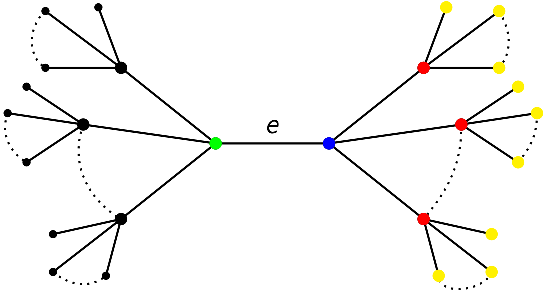
- This notion was introduced by Fouquet and Jolivet (1983).

The Erdős-Nešetřil Conjecture

- The greedy algorithm gives an upper bound $2\Delta(\Delta - 1) + 1$.

Proof.

- Suppose we fix a maximum degree Δ for a graph G. Pick some edge e in the graph G.



- So, the graph has $2((\Delta-1)(\Delta-1)+\Delta-1)+1$
$= 2\Delta(\Delta-1)+1$ total colors.

□

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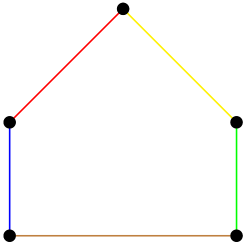

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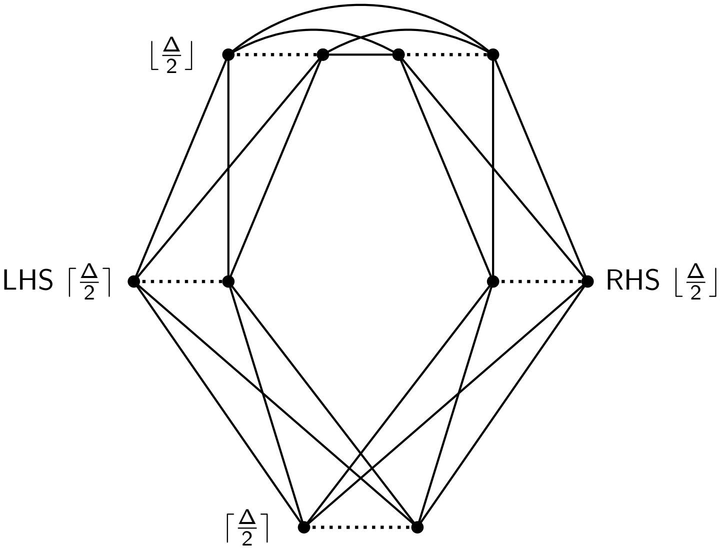

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- For $\Delta = 2$, the bound is 5 colors, and C_5 needs 5 colors.
- For arbitrary Δ, the strong chromatic index has a lower bound given by the C_5 blow-up construction, in which every edge needs a distinct color.
- Conjecture: (Erdős and Nešetřil, 1985) If G is a simple graph with maximum degree Δ, then the C_5 blow-up has the largest strong chromatic index.

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- Therefore, we restrict to considering specific families of graphs, such as planar graphs.

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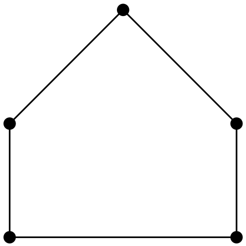

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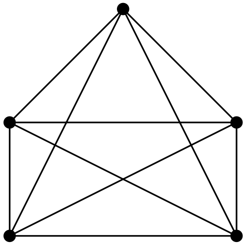
  An example of this is the graph known as K_5.



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  - The embedding of a planar graph into the plane is called its plane presentation.

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- Our best known construction for $\Delta = 4$ requires 13 colors, and these graphs satisfy an alternating condition on the degrees of their vertices.
- For $\Delta = 5$, no good characterization is known.
- Theorem (Nelson and Yu): Let G be a planar graph with a maximum degree of 5. Suppose that for any two vertices v, w if v and w are adjacent, then $\delta(v) + \delta(w) \leq 8$. Then, G has a coloring in 20 colors.

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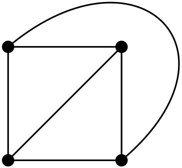

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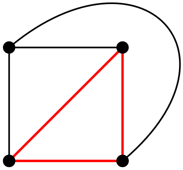

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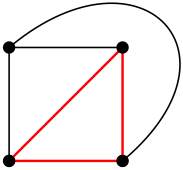

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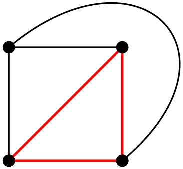

- Every plane presentation of a plane graph has the same number of faces.
- Theorem (Euler): If G is a plane graph with v vertices, e edges, and f faces, then $v - e + f = 2$.

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- Pick an induced subgraph H of G. By minimality, $G - H$ has a coloring. Then, extend the coloring of $G - H$ to H in G.
- If we do this for enough subgraphs H, we can show that if G is planar it violates Euler's Polyhedral Formula $v - e + f = 2$.

Future Questions

- What bound can we obtain more generally in the $\Delta = 5$ case?

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- What lower bound can we obtain for $\Delta = 5$?

Thank you!